CONVEX SETS AND QUANTITIES

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Abstract

The paper emphasizes the importance of convex sets in the study and
application of scalar quantities. Considerations are done in our three-
dimensional space, and the results are valid in the Euclidean space of any
dimension. Some observations can be realized with general measures.

1. Introduction

Let $\mathcal{X}$ be a real vector space. A linear combination of vectors $x_i \in \mathcal{X}$
and coefficients $\alpha_i \in \mathbb{R}$,

$$
\sum_{i=1}^{n} \alpha_i x_i,
$$

is called the convex combination if all $\alpha_i \in [0, 1]$ and $\sum_{i=1}^{n} \alpha_i = 1$. Any
convex combination with at least two positive coefficients can be
expressed in the binomial form, that is,
where $1 \leq k \leq n-1$, $\alpha = \sum_{i=1}^{k} \alpha_i > 0$, and $\beta = \sum_{i=k+1}^{n} \alpha_i > 0$. A set $\mathcal{C} \subseteq \mathcal{X}$ is convex if for any two vectors $x, y \in \mathcal{C}$ all corresponding binomial convex combinations $\alpha x + \beta y$ belong to $\mathcal{C}$.

Let $\mathcal{A} \subseteq \mathcal{X}$ be a set. A convex hull $\text{conv} \mathcal{A}$ of the set $\mathcal{A}$ is the smallest convex set containing $\mathcal{A}$. The convex hull $\text{conv} \mathcal{A}$ consists of all binomial convex combinations of the vectors of $\mathcal{A}$.

The importance of convex analysis can be found in many books, for example, in [2] and [5].

2. Quantity Center as Convex Combination

We observe the problems associated with convex sets in the space $\mathbb{R}^3$. The basic convex sets are segments of the line, convex polygons in the plane, and convex polyhedra in the space. Line segments are bounded with two end-points as extreme points. Convex polygons are plane figures bounded with line segments as edges, and vertices as extreme points. Convex polyhedra are space bodies bounded with convex polygons as faces, and vertices as extreme points.

The radius-vector of a given point $P$ will be denoted by $\vec{r}_P$, and radius-vectors of given points $P_i$ will be denoted by $\vec{r}_i$.

We present the physical meaning of convex combinations. Consider a set of $n$ particles in the space. The value of a certain physical scalar quantity $q$ is associated to each particle of the observed set. Let the particles be located at the points $P_1, \ldots, P_n$. We want to specify the position of the quantity center $Q$. 
Suppose that the particles $P_i$ have quantity values $q_i$ that are non-negative, with positive quantity total value

$$q_{tot} = \sum_{i=1}^{n} q_i.$$ 

Thus, the relative quantity values $p_i = q_i / q_{tot}$ are non-negative with $\sum_{i=1}^{n} p_i = 1$. The quantity arithmetic mean $\bar{q}$ is the arithmetic mean of all the scalars $q_i$, that is,

$$\bar{q} = \frac{q_{tot}}{n} = \frac{1}{n} \sum_{i=1}^{n} q_i. \quad (2.1)$$

The center $Q$ of the quantity $q$ on the set $\{P_1, \ldots, P_n\}$ is defined by the radius-vector equation

$$\sum_{i=1}^{n} q_i (\tilde{r}_i - \tilde{r}_Q) = \tilde{0}. \quad (2.2)$$

As a consequence of the above definition, we have the following well-known representation:

**Theorem A (Quantity center).** Let $P_1, \ldots, P_n \in \mathbb{R}^3$ be points with associated non-negative quantity values $q(P_i) = q_i$ so that $q_{tot} = \sum_{i=1}^{n} q_i > 0$.

If $Q$ is the center of the quantity $q$ on the set $\{P_1, \ldots, P_n\}$, then its radius-vector $\tilde{r}_Q$ is the convex combination of the radius-vectors $\tilde{r}_i$, precisely,

$$\tilde{r}_Q = \frac{1}{q_{tot}} \sum_{i=1}^{n} q_i \tilde{r}_i. \quad (2.3)$$

Consequently, the center $Q$ belongs to the convex hull of the set $\{P_1, \ldots, P_n\}$. 
If \( q \) represents the masses of the particles \( P_i \), then the expression in (2.3) represents the radius-vector of the mass center \( Q \) on the set \( \{P_1, ..., P_n\} \). If \( q = 1 \), then the expression in (2.3) presents the radius-vector of the geometric center \( Q \) of the set \( \{P_1, ..., P_n\} \).

Now, we observe two quantities \( a \) and \( b \) at points \( P_i \). Let \( a(P_i) = a_i \) and \( b(P_i) = b_i \) be respecting quantity values. Let \( A, B, \) and \( Q \) be the centers of the quantities \( a, b, \) and \( q = a + b \), respectively. According to the formula in (2.3), the radius-vector \( \vec{r}_Q \) can be expressed as

\[
\vec{r}_Q = \frac{1}{q_{\text{tot}}} \sum_{i=1}^{n} q_i \vec{r}_i = \frac{1}{a_{\text{tot}} + b_{\text{tot}}} \sum_{i=1}^{n} (a_i + b_i) \vec{r}_i \tag{2.4}
\]

\[
= \frac{a_{\text{tot}}}{a_{\text{tot}} + b_{\text{tot}}} \sum_{i=1}^{n} \frac{a_i}{a_{\text{tot}}} \vec{r}_i + \frac{b_{\text{tot}}}{a_{\text{tot}} + b_{\text{tot}}} \sum_{i=1}^{n} \frac{b_i}{b_{\text{tot}}} \vec{r}_i 
\]

\[
= \frac{a_{\text{tot}}}{a_{\text{tot}} + b_{\text{tot}}} \vec{r}_A + \frac{b_{\text{tot}}}{a_{\text{tot}} + b_{\text{tot}}} \vec{r}_B. \tag{2.5}
\]

The formula in (2.5) can be easily generalized to the sum of several quantities.

**Corollary 2.1 (Quantity sum center).** Let \( P_1, ..., P_n \in \mathbb{R}^3 \) be points with associated non-negative quantity values \( q_j(P_i) = q_{ji} \) so that \( (q_j)_{\text{tot}} = \sum_{i=1}^{n} q_{ji} > 0 \) for \( j = 1, ..., m \). Let \( q = \sum_{j=1}^{m} q_j \) be the sum of all quantities. Let \( Q_j \) be the center of the quantity \( q_j \) on the set \( \{P_1, ..., P_n\} \) for \( j = 1, ..., m \).

If \( Q \) is the center of the quantity sum \( q \) on the set \( \{P_1, ..., P_n\} \), then its radius-vector \( \vec{r}_Q \) is the convex combination of the radius-vectors \( \vec{r}_{Q_j} \), precisely,
\[ \tilde{r}_Q = \frac{1}{q_{\text{tot}}} \sum_{j=1}^{m} (q_j)_{\text{tot}} \tilde{r}_j. \]

Consequently, the center \( Q \) belongs to the convex hull of the set \( \{P_1, \ldots, P_n\} \).

Complete yet the geometric and analytic picture separating the given set of points in the two parts
\[ \{P_1, \ldots, P_n\} = \{P_1, \ldots, P_k\} \cup \{P_{k+1}, \ldots, P_n\}. \]

Applying the formula in (2.1) for the quantity arithmetic mean \( \tilde{q} \), and presenting it as the binomial “scalar convex combination” of the separated parts, it follows:
\[ \tilde{q} = \sum_{i=1}^{n} \frac{1}{n} q_i = \frac{k}{n} \tilde{q}_k + \frac{n-k}{n} \tilde{q}_{n-k}, \]
where
\[ \tilde{q}_k = \sum_{i=1}^{k} \frac{1}{k} q_i, \quad \tilde{q}_{n-k} = \sum_{i=k+1}^{n} \frac{1}{n-k} q_i. \]

Thus, the quantity arithmetic mean \( \tilde{q} \) is the binomial “scalar convex combination” of the \( k \)-membered “scalar convex combination” \( \tilde{q}_k \) and \( (n-k) \)-membered “scalar convex combination” \( \tilde{q}_{n-k} \).

Let \( A \) be the quantity center on the set \( \{P_1, \ldots, P_k\} \), and \( B \) be the quantity center on the set \( \{P_{k+1}, \ldots, P_n\} \). Using the formula in (2.3) for the radius-vector \( \tilde{r}_Q \) of the quantity center \( Q \) on the whole set \( \{P_1, \ldots, P_n\} \), and presenting it as the binomial convex combination of the subcenter radius-vectors \( \tilde{r}_A \) and \( \tilde{r}_B \), it follows:
\[ \tilde{r}_Q = \frac{1}{q_{\text{tot}}} \sum_{i=1}^{n} q_i \tilde{r}_i = \alpha \tilde{r}_A + \beta \tilde{r}_B, \]
where
\[
\alpha = \frac{1}{q_{\text{tot}}} \sum_{i=1}^{k} q_i, \quad \beta = \frac{1}{q_{\text{tot}}} \sum_{i=k+1}^{n} q_i, \quad (2.10)
\]
and
\[
\bar{r}_A = \frac{1}{\alpha q_{\text{tot}}} \sum_{i=1}^{k} q_i \bar{r}_i, \quad \bar{r}_B = \frac{1}{\beta q_{\text{tot}}} \sum_{i=k+1}^{n} q_i \bar{r}_i. \quad (2.11)
\]
We find that the radius-vector \( \bar{r}_Q \) is the binomial convex combination of the \( k \)-membered convex combination \( \bar{r}_A \) and \( (n-k) \)-membered convex combination \( \bar{r}_B \). This representation can be generalized.

**Corollary 2.2 (Quantity subcenters).** Let \( P_1, \ldots, P_n \in \mathbb{R}^3 \) be points with associated non-negative quantity values \( q(P_i) = q_i \). Let
\[
\{P_1, \ldots, P_n\} = \bigcup_{j=1}^{m} \{P_{k_{j-1}+1}, \ldots, P_{k_{j-1}+k_j}\},
\]
be partition of the set of points \( P_i \), where \( k_0 = 0 \) and \( \sum_{j=1}^{m} k_j = n \).

If \( Q \) is the quantity center on the set \( \{P_1, \ldots, P_n\} \), and if \( Q_j \) is the quantity center on the set \( \{P_{k_{j-1}+1}, \ldots, P_{k_{j-1}+k_j}\} \) for \( j = 1, \ldots, m \), then
\[
\bar{r}_Q = \sum_{j=1}^{m} \alpha_j \bar{r}_{Q_j}, \quad (2.12)
\]
where
\[
\alpha_j = \frac{1}{q_{\text{tot}}} \sum_{i=k_{j-1}+1}^{k_{j-1}+k_j} q_i > 0, \quad \bar{r}_{Q_j} = \frac{1}{\alpha_j q_{\text{tot}}} \sum_{i=k_{j-1}+1}^{k_{j-1}+k_j} q_i \bar{r}_i. \quad (2.13)
\]
We have the following two examples:

**Example 2.3 (Quantity subcenters).** Let \( P_1(0, 0, 0), P_2(3, 0, 0), P_3(0, 6, 0), P_4(0, 4, 3), P_5(0, 0, 3), P_6(0, 4, 0), \) and \( P_7(1, 0, 1) \) be points with the associated quantity values \( q_i = q(P_i) = x_i + y_i + z_i. \) Note that the points \( P_1 - P_5 \) are the vertices of the convex pentahedron, and the points \( P_6 - P_7 \) belong to this pentahedron, as shown in Figure 1. Should be expressed the quantity center \( Q \) on the set \( \{ P_1, \ldots, P_7 \} \) using the quantity center \( A \) on the set \( \{ P_1, P_2, P_3, P_4, P_5 \} \), and the quantity center \( B \) on the set \( \{ P_6, P_7 \} \).

![Figure 1. Convex pentahedron of Example 2.3.](image)

The radius-vector \( \vec{r}_Q \), according to the expressions in (2.9)-(2.11), can be written as follows:

\[
\vec{r}_Q = \frac{1}{q_{tot}} \sum_{i=1}^{7} q_i \vec{r}_i = \frac{11}{25} \vec{i} + \frac{80}{25} \vec{j} + \frac{32}{25} \vec{k}
\]

\[
= \frac{19}{25} \vec{r}_A + \frac{6}{25} \vec{r}_B,
\]
where

\[
\vec{r}_A = \frac{1}{19} \sum_{i=1}^{5} \frac{q_i}{q_{\text{tot}}} \vec{r}_i = \frac{9}{19} \vec{i} + \frac{64}{19} \vec{j} + \frac{30}{19} \vec{k},
\]

and

\[
\vec{r}_B = \frac{1}{6} \sum_{i=6}^{7} \frac{q_i}{q_{\text{tot}}} \vec{r}_i = \frac{1}{3} \vec{i} + \frac{8}{3} \vec{j} + \frac{1}{3} \vec{k}.
\]

**Example 2.4 (Quantity influence).** Let \( P_1, \ldots, P_n \in \mathbb{R}^3 \) be points with associated quantity values \( q(P_i) = q_i \). Should be approximately determined the influence \( \vec{q} \) of the quantity \( q \) at the point \( P \in \mathbb{R}^3 \).

Suppose that \( Q \) is the center of the quantity \( q \) on the set \( \{P_1, \ldots, P_n\} \), and take the vector \( \vec{v} = Q \vec{P} \). For every \( i = 1, \ldots, n \), let \( P_{iv} \) be the end-point of the radius-vector \( \vec{r}_{iv} = \vec{r}_i + \vec{v} \), and \( \vec{q}(P_{iv}) = \vec{q}_{iv} \) be the influence of \( q_i \) from \( P_i \). Then the convex combination

\[
\vec{r}_P = \vec{r}_Q + \vec{v} = \frac{1}{q_{\text{tot}}} \sum_{i=1}^{n} q_i \vec{r}_i + \vec{v} = \frac{1}{q_{\text{tot}}} \sum_{i=1}^{n} q_i (\vec{r}_i + \vec{v}) \approx \frac{1}{q_{\text{tot}}} \sum_{i=1}^{n} \vec{q}_{iv} \vec{r}_{iv},
\]

indicates that the point \( P \) may be treated as the center of the quantity influence \( \vec{q} \) on the set \( \{P_{1v}, \ldots, P_{nv}\} \). Therefore, the influence of the quantity \( q \) at the point \( P \) is the arithmetic mean of all the quantity influences \( \vec{q}_{iv} \), that is,

\[
\vec{q}(P) = \overline{\vec{q}} = \frac{1}{n} \sum_{i=1}^{n} \vec{q}_{iv}.
\]

### 3. Quantity Barycenter as Integral

This section presents the main results of the article.

Applying the integral method to a sequence of convex combinations, we get the integral, so the center of quantity passes to the barycenter of
quantity. Some results on barycentres and integral arithmetic means for the subsets of the line were obtained in [3] and [4].

Continue with observations in the space where the particles are not separate, but form a continuous set. Let \( A \subset \mathbb{R}^3 \) be a set with positive volume \( \text{vol}(A) \), and \( q : A \to \mathbb{R} \) be a non-negative quantity (scalar field) as the Riemann integrable function of three variables with positive \( \int_A \int_A \int_A q(x, y, z)dx dy dz \). First, we imitate the previous discrete case. Given a positive integer \( n \), it is necessary to make the partition of the set \( A \) into the union

\[
A = \bigcup_{i=1}^{n} A_{ni},
\]

of pairwise disjoint sets \( A_{ni} \) with positive volumes \( \text{vol}(A_{ni}) \), where every \( A_{ni} \) contracts to the point or vanishes in infinity as \( n \) goes to infinity. For every \( i = 1, \ldots, n \), we take the one point \( P_{ni} \in A_{ni} \), and its quantity value \( q_{ni} = q(P_{ni}) \). Use the following adaptations:

\[
q_{ni} \approx \frac{n}{\text{vol}(A)} \text{vol}(A_{ni}) q_{ni}, \quad q_{tot} \approx \frac{n}{\text{vol}(A)} \sum_{i=1}^{n} \text{vol}(A_{ni}) q_{ni}.
\]

Respecting the formulas in (2.1) and (2.3), we have the discrete approximations

\[
\bar{q} \approx \frac{1}{\text{vol}(A)} \sum_{i=1}^{n} \text{vol}(A_{ni}) q_{ni}, \quad (3.1)
\]

and

\[
\bar{r}Q \approx \frac{1}{\sum_{i=1}^{n} \text{vol}(A_{ni}) q_{ni}} \sum_{i=1}^{n} \text{vol}(A_{ni}) q_{ni} \bar{r}_{ni}. \quad (3.2)
\]

Letting \( n \) to infinity, the above discrete approximations get the Riemann integral forms as follows. Put \( \bar{r}(x, y, z) = x\bar{i} + y\bar{j} + z\bar{k} \). The number
\[
\bar{q} = \frac{1}{\text{vol}(A)} \iiint_A q(x, y, z) dx\,dy\,dz, \quad (3.3)
\]
is the integral arithmetic mean of the quantity \(q(x, y, z)\) on the set \(A\).

The radius-vector
\[
\bar{r}_Q = \frac{1}{\iiint_A q(x, y, z) dx\,dy\,dz} \iiint_A \bar{r}(x, y, z) q(x, y, z) dx\,dy\,dz, \quad (3.4)
\]
presents the barycenter \(Q\) of the quantity \(q(x, y, z)\) on the set \(A\). The longer form is
\[
\iiint_A \bar{r}(x, y, z) q(x, y, z) dx\,dy\,dz = \iiint_A xq(x, y, z) dx\,dy\,dz
\]
\[
+ \iiint_A yq(x, y, z) dx\,dy\,dz
\]
\[
+ \iiint_A zq(x, y, z) dx\,dy\,dz.
\]

If we take \(q = 1\), the formula in (3.4) represents the barycenter \(Q\) of the set \(A\),
\[
\bar{r}_Q = \frac{1}{\text{vol}(A)} \iiint_A \bar{r}(x, y, z) dx\,dy\,dz. \quad (3.5)
\]
The known role for set barycenters is as follows.

**Lemma 3.1.** Let \(A = \bigcup_{i=1}^n A_i\) be a union of pairwise disjoint sets \(A_i \subset \mathbb{R}^3\) with \(\text{vol}(A_i) > 0\).

If \(Q\) is the barycenter of \(A\), and \(Q_i\) are the barycenters of \(A_i\), then
\[
\bar{r}_Q = \frac{1}{\text{vol}(A)} \sum_{i=1}^n \frac{\text{vol}(A_i)}{\text{vol}(A)} \bar{r}_{Q_i}. \quad (3.6)
\]
If \( q(x, y, z) \) represents the mass density of the spatial body \( \mathcal{A} \), then the expression in (3.4) represents the radius-vector of the mass density barycenter \( Q \) of the body \( \mathcal{A} \).

If \( \mathcal{A} \) is connected and \( q(x, y, z) \) is continuous, then the integral arithmetic mean \( \overline{q} \) belongs to the set \( q(\mathcal{A}) \). From the convex combination formula in (3.2), it can be concluded that the barycenter \( Q \) of the quantity \( q \) on the set \( \mathcal{A} \) belongs to the convex hull \( \text{conv} \mathcal{A} \) of the set \( \mathcal{A} \). The formula in (3.4) arises after taking the limit, so the barycenter \( Q \) belongs to the closed convex hull of the set \( \mathcal{A} \).

The geometrical description of convexity will be applied to prove the following theorem. A similar approach was used in [1] for convex polygons.

**Theorem 3.2.** Let \( \mathcal{A} \subset \mathbb{R}^3 \) be a set such that \( \text{vol}(\mathcal{A}) > 0 \), and let \( q : \mathcal{A} \to \mathbb{R} \) be an integrable (the Riemann integrable) quantity function such that \( q(x, y, z) > 0 \) for every point \( (x, y, z) \in \text{int} \mathcal{A} \).

Then the \( \mu \)-barycenter \( \mu Q \) of the quantity \( q \) on the set \( \mathcal{A} \) belongs to the interior of the convex hull of the set \( \mathcal{A} \), that is, \( Q \in \text{int}\{\text{conv} \mathcal{A}\} \).

**Proof.** Assume that \( \mathcal{A} \) is convex, and prove that \( Q \in \text{int} \mathcal{A} \).

The geometrical characterization of convexity of the set \( \mathcal{A} \) with \( \text{int} \mathcal{A} \neq \emptyset \) states that the plane \( \mathcal{P} \) through each point outside of \( \text{int} \mathcal{A} \) exists so that \( \mathcal{P} \cap \text{int} \mathcal{A} = \emptyset \), and the entire \( \text{int} \mathcal{A} \) is contained in one of the half-spaces determined by \( \mathcal{P} \).

Suppose \( Q \notin \text{int} \mathcal{A} \), and take the characteristic plane \( \mathcal{P} \) passing through the point \( Q \). Let \( \vec{n} \) be the vector normal to the plane \( \mathcal{P} \), oriented towards the set \( \mathcal{A} \) as shown in Figure 2.
Take abbreviations $q = q(x, y, z)$ and $\bar{r} = \bar{r}(x, y, z)$. Using the inner product with vector $\vec{n}$, we find that the following holds for every $(x, y) \in \text{int} \ A$:

$$\langle \vec{n}, \bar{r} - \bar{r}_Q \rangle > 0,$$

$$\langle \vec{n}, \bar{r} - \frac{1}{\int \int \int_A q \, dxdydz} \int \int \int_A \bar{r} q \, dxdydz \rangle > 0,$$

$$\langle \vec{n}, \bar{r}_Q \rangle - \frac{q}{\int \int \int_A q \, dxdydz} \langle \vec{n}, \int \int \int_A \bar{r} q \, dxdydz \rangle > 0.$$

Integrating the above inequality over $\text{int} \ A$, it follows that

$$\langle \vec{n}, \int \int \int_A \bar{r} q \, dxdydz \rangle - \langle \vec{n}, \int \int \int_A \bar{r} q \, dxdydz \rangle > 0,$$

because the integrals over $\text{int} \ A$ and $A$ are the same. The resulting contradiction says that must be $Q \in \text{int} \ A$. \qed
4. Generalizations to Higher Dimensions

Using the general theory of measure and integral, we will show that Theorem 3.2 is valid in the Euclidean space of any dimension.

**Lemma 4.1.** Let $\mu$ be a measure on $\mathbb{R}^n$, let $\mathcal{A}$ be a $\mu$-measurable set such that $\mu(\text{int}\mathcal{A}) > 0$.

Then the $\mu$-barycenter $Q$ of the set $\mathcal{A}$ belongs to the interior of the convex hull of the set $\mathcal{A}$, that is, $Q \in \text{int}\{\text{conv}\mathcal{A}\}$.

**Proof.** Without loss of generality, we can suppose that the set $\mathcal{A}$ is convex, and prove $Q \in \text{int}\mathcal{A}$. Put $\mathcal{A}_1 = \text{int}\mathcal{A}$, $\mathcal{A}_2 = \mathcal{A} \setminus \mathcal{A}_1$, $Q_1$ is the $\mu$-barycenter of $\mathcal{A}_1$, and $Q_2$ is the $\mu$-barycenter of $\mathcal{A}_2$. We have two cases respecting the measure of the set $\mathcal{A}_2$.

If $\mu(\mathcal{A}_2) = 0$, it is obvious that $Q \in \mathcal{A}_1 = \text{int}\mathcal{A}$.

Assume that $\mu(\mathcal{A}_2) > 0$. If $Q_1 = Q_2$, the assertion of lemma is trivially satisfied. If $Q_1 \neq Q_2$, then the convex combination

$$\bar{r}_Q = \frac{\mu(\mathcal{A}_1)}{\mu(\mathcal{A})} \bar{r}_{Q_1} + \frac{\mu(\mathcal{A}_2)}{\mu(\mathcal{A})} \bar{r}_{Q_2},$$

(4.1)

has positive coefficients $\alpha_1 = \mu(\mathcal{A}_1)/\mu(\mathcal{A})$ and $\alpha_2 = \mu(\mathcal{A}_2)/\mu(\mathcal{A})$. Therefore, in this case, the barycenter $Q$ belongs to the open line segment $(Q_1, Q_2)$, so it follows that $Q \in (Q_1, Q_2) \subseteq \mathcal{A}_1 = \text{int}\mathcal{A}_1$. \qed

**Theorem 4.2.** Let $\mu$ be measure on $\mathbb{R}^n$, let $\mathcal{A}$ be a $\mu$-measurable set such that $\mu(\text{int}\mathcal{A}) > 0$, and let $q : \mathcal{A} \to \mathbb{R}$ be a $\mu$-integrable (the Lebesgue integrable) quantity function such that $q(P) > 0$ for every point $P \in \text{int}\mathcal{A}$.

Then the $\mu$-barycenter $Q$ of the quantity $q$ on the set $\mathcal{A}$ belongs to the interior of the convex hull of the set $\mathcal{A}$. 
Proof. We define the new measure $\nu$ of the $\mu$-measurable set $A$ by

$$\nu(A) = \int_A q \, d\mu. \quad (4.2)$$

Let $R$ be the $\nu$-barycenter of the set $A$,

$$\bar{r}_R = \frac{1}{\nu(A)} \int_A \bar{r} \, d\nu. \quad (4.3)$$

Since $\nu(\text{int} \, A) > 0$, it follows that $R \in \text{int}\{\text{conv} \, A\}$ by Lemma 4.1.

Applying Equations (4.2) and (4.3), we get the formula

$$\bar{r}_Q = \frac{1}{\nu(A)} \int_A \bar{r}q \, d\mu = \frac{1}{\nu(A)} \int_A \bar{r} \, d\nu = \bar{r}_R. \quad (4.4)$$

and reach the conclusion $Q = R$, so $Q \in \text{int}\{\text{conv} \, A\}$. \qed

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References


